Mathematics for College Technology, Grade 12

College Preparation **MCT4C**

This course enables students to extend their knowledge of functions. Students will investigate and apply properties of polynomial, exponential, and trigonometric functions; continue to represent functions numerically, graphically, and algebraically; develop facility in simplifying expressions and solving equations; and solve problems that address applications of algebra, trigonometry, vectors, and geometry. Students will reason mathematically and communicate their thinking as they solve multi-step problems. This course prepares students for a variety of college technology programs.

Prerequisite: Functions and Applications, Grade 11, University/College Preparation, or Functions, Grade 11, University Preparation

MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

Problem Solving	• develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
Reasoning and Proving	• develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
Reflecting	• demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
Selecting Tools and Computational Strategies	• select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
Connecting	• make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
Representing	• create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
Communicating	• communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

A. EXPONENTIAL FUNCTIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. solve problems involving exponential equations graphically, including problems arising from real-world applications;

2. solve problems involving exponential equations algebraically using common bases and logarithms, including problems arising from real-world applications.

SPECIFIC EXPECTATIONS

1. Solving Exponential Equations Graphically

By the end of this course, students will:

1.1 determine, through investigation with technology, and describe the impact of changing the base and changing the sign of the exponent on the graph of an exponential function

1.2 solve simple exponential equations numerically and graphically, with technology (e.g., use systematic trial with a scientific calculator to determine the solution to the equation $1.05^x = 1{,}276$), and recognize that the solutions may not be exact

Sample problem: Use the graph of $y = 3^x$ to solve the equation $3^x = 5$.

1.3 determine, through investigation using graphing technology, the point of intersection of the graphs of two exponential functions (e.g., $y = 4^{-x}$ and $y = 8^{x+3}$), recognize the x-coordinate of this point to be the solution to the corresponding exponential equation (e.g., $4^{-x} = 8^{x+3}$), and solve exponential equations graphically (e.g., solve $2^{x+2} = 2^x + 12$ by using the intersection of the graphs of $y = 2^{x+2}$ and $y = 2^x + 12$)

Sample problem: Solve $0.5^x = 3^{x+3}$ graphically.

1.4 pose problems based on real-world applications (e.g., compound interest, population growth) that can be modelled with exponential equations, and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

Sample problem: A tire with a slow puncture loses pressure at the rate of 4%/min. If the tire's pressure is 300 kPa to begin with, what is its pressure after 1 min? After 2 min? After 10 min? Use graphing technology to determine when the tire's pressure will be 200 kPa.

2. Solving Exponential Equations Algebraically

By the end of this course, students will:

2.1 simplify algebraic expressions containing integer and rational exponents using the laws of exponents (e.g., $x^3 \div x^{\frac{1}{2}}$, $\sqrt{x^6y^{12}}$)

Sample problem: Simplify $\dfrac{a^3b^2c^3}{\sqrt{a^2b^4}}$ and then evaluate for $a = 4$, $b = 9$, and $c = -3$. Verify your answer by evaluating the expression without simplifying first. Which method for evaluating the expression do you prefer? Explain.

2.2 solve exponential equations in one variable by determining a common base (e.g., $2^x = 32$, $4^{5x-1} = 2^{2(x+11)}$, $3^{5x+8} = 27^x$)

Sample problem: Solve $3^{5x+8} = 27^x$ by determining a common base, verify by substitution, and investigate connections to the intersection of $y = 3^{5x+8}$ and $y = 27^x$ using graphing technology.

2.3 recognize the logarithm of a number to a given base as the exponent to which the base must be raised to get the number, recognize the operation of finding the logarithm to be the inverse operation (i.e., the undoing or reversing) of exponentiation, and evaluate simple logarithmic expressions

Sample problem: Why is it possible to determine $\log_{10}(100)$ but not $\log_{10}(0)$ or $\log_{10}(-100)$? Explain your reasoning.

2.4 determine, with technology, the approximate logarithm of a number to any base, including base 10 [e.g., by recognizing that $\log_{10}(0.372)$ can be determined using the LOG key on a calculator; by reasoning that $\log_3 29$ is between 3 and 4 and using systematic trial to determine that $\log_3 29$ is approximately 3.07]

2.5 make connections between related logarithmic and exponential equations (e.g., $\log_5 125 = 3$ can also be expressed as $5^3 = 125$), and solve simple exponential equations by rewriting them in logarithmic form (e.g., solving $3^x = 10$ by rewriting the equation as $\log_3 10 = x$)

2.6 pose problems based on real-world applications that can be modelled with given exponential equations, and solve these and other such problems algebraically by rewriting them in logarithmic form

Sample problem: When a potato whose temperature is 20°C is placed in an oven maintained at 200°C, the relationship between the core temperature of the potato T, in degrees Celsius, and the cooking time t, in minutes, is modelled by the equation $200 - T = 180(0.96)^t$. Use logarithms to determine the time when the potato's core temperature reaches 160°C.

B. POLYNOMIAL FUNCTIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. recognize and evaluate polynomial functions, describe key features of their graphs, and solve problems using graphs of polynomial functions;

2. make connections between the numeric, graphical, and algebraic representations of polynomial functions;

3. solve polynomial equations by factoring, make connections between functions and formulas, and solve problems involving polynomial expressions arising from a variety of applications.

SPECIFIC EXPECTATIONS

1. Investigating Graphs of Polynomial Functions

By the end of this course, students will:

1.1 recognize a polynomial expression (i.e., a series of terms where each term is the product of a constant and a power of x with a non-negative integral exponent, such as $x^3 - 5x^2 + 2x - 1$); recognize the equation of a polynomial function and give reasons why it is a function, and identify linear and quadratic functions as examples of polynomial functions

1.2 compare, through investigation using graphing technology, the graphical and algebraic representations of polynomial (i.e., linear, quadratic, cubic, quartic) functions (e.g., investigate the effect of the degree of a polynomial function on the shape of its graph and the maximum number of x-intercepts; investigate the effect of varying the sign of the leading coefficient on the end behaviour of the function for very large positive or negative x-values)

Sample problem: Investigate the maximum number of x-intercepts for linear, quadratic, cubic, and quartic functions using graphing technology.

1.3 describe key features of the graphs of polynomial functions (e.g., the domain and range, the shape of the graphs, the end behaviour of the functions for very large positive or negative x-values)

Sample problem: Describe and compare the key features of the graphs of the functions $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, and $f(x) = x^4$.

1.4 distinguish polynomial functions from sinusoidal and exponential functions [e.g., $f(x) = \sin x$, $f(x) = 2^x$)], and compare and contrast the graphs of various polynomial functions with the graphs of other types of functions

1.5 substitute into and evaluate polynomial functions expressed in function notation, including functions arising from real-world applications

Sample problem: A box with no top is being made out of a 20-cm by 30-cm piece of cardboard by cutting equal squares of side length x from the corners and folding up the sides. The volume of the box is $V = x(20 - 2x)(30 - 2x)$. Determine the volume if the side length of each square is 6 cm. Use the graph of the polynomial function $V(x)$ to determine the size of square that should be cut from the corners if the required volume of the box is 1000 cm^3.

1.6 pose problems based on real-world applications that can be modelled with polynomial functions, and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

1.7 recognize, using graphs, the limitations of modelling a real-world relationship using a polynomial function, and identify and explain any restrictions on the domain and range (e.g., restrictions on the height and time for a polynomial function that models the relationship between height above the ground and time for a falling object)

Sample problem: The forces acting on a horizontal support beam in a house cause it to sag by d centimetres, x metres from one end of the beam. The relationship between d and x can be represented by the polynomial function $d(x) = \frac{1}{1850}x(1000 - 20x^2 + x^3)$. Graph the function, using technology, and determine the domain over which the function models the relationship between d and x. Determine the length of the beam using the graph, and explain your reasoning.

2. Connecting Graphs and Equations of Polynomial Functions

By the end of this course, students will:

2.1 factor polynomial expressions in one variable, of degree no higher than four, by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring)

Sample problem: Factor: $x^4 - 16$; $x^3 - 2x^2 - 8x$.

2.2 make connections, through investigation using graphing technology (e.g., dynamic geometry software), between a polynomial function given in factored form [e.g., $f(x) = x(x - 1)(x + 1)$] and the x-intercepts of its graph, and sketch the graph of a polynomial function given in factored form using its key features (e.g., by determining intercepts and end behaviour; by locating positive and negative regions using test values between and on either side of the x-intercepts)

Sample problem: Sketch the graphs of $f(x) = -(x - 1)(x + 2)(x - 4)$ and $g(x) = -(x - 1)(x + 2)(x + 2)$ and compare their shapes and the number of x-intercepts.

2.3 determine, through investigation using technology (e.g., graphing calculator, computer algebra systems), and describe the connection between the real roots of a polynomial equation and the x-intercepts of the graph of the corresponding polynomial function [e.g., the real roots of the equation $x^4 - 13x^2 + 36 = 0$ are the x-intercepts of the graph of $f(x) = x^4 - 13x^2 + 36$]

Sample problem: Describe the relationship between the x-intercepts of the graphs of linear and quadratic functions and the real roots of the corresponding equations. Investigate, using technology, whether this relationship exists for polynomial functions of higher degree.

3. Solving Problems Involving Polynomial Equations

By the end of this course, students will:

3.1 solve polynomial equations in one variable, of degree no higher than four (e.g., $x^2 - 4x = 0$, $x^4 - 16 = 0$, $3x^2 + 5x + 2 = 0$), by selecting and applying strategies (i.e., common factoring; difference of squares; trinomial factoring), and verify solutions using technology (e.g., using computer algebra systems to determine the roots of the equation; using graphing technology to determine the x-intercepts of the corresponding polynomial function)

Sample problem: Solve $x^3 - 2x^2 - 8x = 0$.

3.2 solve problems algebraically that involve polynomial functions and equations of degree no higher than four, including those arising from real-world applications

3.3 identify and explain the roles of constants and variables in a given formula (e.g., a constant can refer to a known initial value or a known fixed rate; a variable changes with varying conditions)

Sample problem: The formula $P = P_0 + kh$ is used to determine the pressure, P kilopascals, at a depth of h metres under water, where k kilopascals per metre is the rate of change of the pressure as the depth increases, and P_0 kilopascals is the pressure at the surface. Identify and describe the roles of P, P_0, k, and h in this relationship, and explain your reasoning.

3.4 expand and simplify polynomial expressions involving more than one variable [e.g., simplify $-2xy(3x^2y^3 - 5x^3y^2)$], including expressions arising from real-world applications

Sample problem: Expand and simplify the expression $\pi(R + r)(R - r)$ to explain why it represents the area of a ring. Draw a diagram of the ring and identify R and r.

3.5 solve equations of the form $x^n = a$ using rational exponents (e.g., solve $x^3 = 7$ by raising both sides to the exponent $\frac{1}{3}$)

3.6 determine the value of a variable of degree no higher than three, using a formula drawn from an application, by first substituting known values and then solving for the variable, and by first isolating the variable and then substituting known values

Sample problem: The formula $s = ut + \frac{1}{2}at^2$ relates the distance, s, travelled by an object to its initial velocity, u, acceleration, a, and the elapsed time, t. Determine the acceleration of a dragster that travels 500 m from rest in 15 s, by first isolating a, and then by first substituting known values. Compare and evaluate the two methods.

3.7 make connections between formulas and linear, quadratic, and exponential functions [e.g., recognize that the compound interest formula, $A = P(1 + i)^n$, is an example of an exponential function $A(n)$ when P and i are constant, and of a linear function $A(P)$ when i and n are constant], using a variety of tools and strategies (e.g., comparing the graphs generated with technology when different variables in a formula are set as constants)

Sample problem: Which variable(s) in the formula $V = \pi r^2 h$ would you need to set as a constant to generate a linear equation? A quadratic equation?

3.8 solve multi-step problems requiring formulas arising from real-world applications (e.g., determining the cost of two coats of paint for a large cylindrical tank)

3.9 gather, interpret, and describe information about applications of mathematical modelling in occupations, and about college programs that explore these applications

C. TRIGONOMETRIC FUNCTIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. determine the values of the trigonometric ratios for angles less than 360º, and solve problems using the primary trigonometric ratios, the sine law, and the cosine law;
2. make connections between the numeric, graphical, and algebraic representations of sinusoidal functions;
3. demonstrate an understanding that sinusoidal functions can be used to model some periodic phenomena, and solve related problems, including those arising from real-world applications.

SPECIFIC EXPECTATIONS

1. Applying Trigonometric Ratios

By the end of this course, students will:

1.1 determine the exact values of the sine, cosine, and tangent of the special angles 0°, 30°, 45°, 60°, 90°, and their multiples

1.2 determine the values of the sine, cosine, and tangent of angles from 0º to 360º, through investigation using a variety of tools (e.g., dynamic geometry software, graphing tools) and strategies (e.g., applying the unit circle; examining angles related to the special angles)

1.3 determine the measures of two angles from 0º to 360º for which the value of a given trigonometric ratio is the same (e.g., determine one angle using a calculator and infer the other angle)

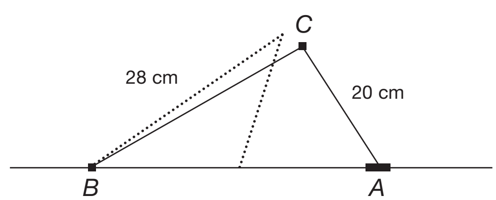
Sample problem: Determine the approximate measures of the angles from 0º to 360º for which the sine is 0.3423.

1.4 solve multi-step problems in two and three dimensions, including those that arise from real-world applications (e.g., surveying, navigation), by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios

Sample problem: Explain how you could find the height of an inaccessible antenna on top of a tall building, using a measuring tape, a clinometer, and trigonometry. What would you measure, and how would you use the data to calculate the height of the antenna?

1.5 solve problems involving oblique triangles, including those that arise from real-world applications, using the sine law (including the ambiguous case) and the cosine law

Sample problem: The following diagram represents a mechanism in which point B is fixed, point C is a pivot, and a slider A can move horizontally as angle B changes. The minimum value of angle B is 35º. How far is it from the extreme left position to the extreme right position of slider A?



2. Connecting Graphs and Equations of Sinusoidal Functions

By the end of this course, students will:

2.1 make connections between the sine ratio and the sine function and between the cosine ratio and the cosine function by graphing the relationship between angles from 0º to 360º and the corresponding sine ratios or cosine ratios, with or without technology (e.g., by generating a table of values using a calculator; by unwrapping the unit circle), defining this relationship as the function $f(x) = \sin x$ or $f(x) = \cos x$, and explaining why the relationship is a function

2.2 sketch the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ for angle measures expressed in degrees, and determine and describe their key properties (i.e., cycle, domain, range, intercepts, amplitude, period, maximum and minimum values, increasing/decreasing intervals)

Sample problem: Describe and compare the key properties of the graphs of $f(x) = \sin x$ and $f(x) = \cos x$. Make some connections between the key properties of the graphs and your understanding of the sine and cosine ratios.

2.3 determine, through investigation using technology, the roles of the parameters d and c in functions of the form $y = \sin(x - d) + c$ and $y = \cos(x - d) + c$, and describe these roles in terms of transformations on the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ with angles expressed in degrees (i.e., vertical and horizontal translations)

Sample problem: Investigate the graph $f(x) = 2\sin(x - d) + 10$ for various values of d, using technology, and describe the effects of changing d in terms of a transformation.

2.4 determine, through investigation using technology, the roles of the parameters a and k in functions of the form $y = a\sin kx$ and $y = a\cos kx$, and describe these roles in terms of transformations on the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ with angles expressed in degrees (i.e., reflections in the axes; vertical and horizontal stretches and compressions to and from the x- and y-axes)

Sample problem: Investigate the graph $f(x) = 2\sin kx$ for various values of k, using technology, and describe the effects of changing k in terms of transformations.

2.5 determine the amplitude, period, and phase shift of sinusoidal functions whose equations are given in the form $f(x) = a\sin(k(x - d)) + c$ or $f(x) = a\cos(k(x - d)) + c$, and sketch graphs of $y = a\sin(k(x - d)) + c$ and $y = a\cos(k(x - d)) + c$ by applying transformations to the graphs of $f(x) = \sin x$ and $f(x) = \cos x$

Sample problem: Transform the graph of $f(x) = \cos x$ to sketch $g(x) = 3\cos(x + 90°)$ and $h(x) = \cos(2x) - 1$, and state the amplitude, period, and phase shift of each function.

2.6 represent a sinusoidal function with an equation, given its graph or its properties

Sample problem: A sinusoidal function has an amplitude of 2 units, a period of 180º, and a maximum at (0, 3). Represent the function with an equation in two different ways, using first the sine function and then the cosine function.

3. Solving Problems Involving Sinusoidal Functions

By the end of this course, students will:

3.1 collect data that can be modelled as a sinusoidal function (e.g., voltage in an AC circuit, pressure in sound waves, the height of a tack on a bicycle wheel that is rotating at a fixed speed), through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials, measurement tools such as motion sensors), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

Sample problem: Measure and record distance–time data for a swinging pendulum, using a motion sensor or other measurement tools, and graph the data. Describe how the graph would change if you moved the pendulum further away from the motion sensor. What would you do to generate a graph with a smaller amplitude?

3.2 identify periodic and sinusoidal functions, including those that arise from real-world applications involving periodic phenomena, given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range

Sample problem: The depth, w metres, of water in a lake can be modelled by the function $w = 5\sin(31.5n + 63) + 12$, where n is the number of months since January 1, 1995. Identify and explain the restrictions on the domain and range of this function.

3.3 pose problems based on applications involving a sinusoidal function, and solve these and other such problems by using a given graph or a graph generated with technology, in degree mode, from a table of values or from its equation

Sample problem: The height above the ground of a rider on a Ferris wheel can be modelled by the sinusoidal function $h(t) = 25\cos(3(t - 60)) + 27$, where $h(t)$ is the height in metres and t is the time in seconds. Graph the function, using graphing technology in degree mode, and determine the maximum and minimum heights of the rider, the height after 30 s, and the time required to complete one revolution.

D. APPLICATIONS OF GEOMETRY

OVERALL EXPECTATIONS

By the end of this course, students will:

1. represent vectors, add and subtract vectors, and solve problems using vector models, including those arising from real-world applications;
2. solve problems involving two-dimensional shapes and three-dimensional figures and arising from real-world applications;
3. determine circle properties and solve related problems, including those arising from real-world applications.

SPECIFIC EXPECTATIONS

1. Modelling With Vectors

By the end of this course, students will:

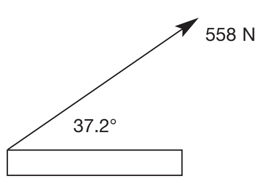
1.1 recognize a vector as a quantity with both magnitude and direction, and identify, gather, and interpret information about real-world applications of vectors (e.g., displacement; forces involved in structural design; simple animation of computer graphics; velocity determined using GPS)

Sample problem: Position is represented using vectors. Explain why knowing that someone is 69 km from Lindsay, Ontario, is not sufficient to identify their exact position.

1.2 represent a vector as a directed line segment, with directions expressed in different ways (e.g., 320°; N 40° W), and recognize vectors with the same magnitude and direction but different positions as equal vectors

1.3 resolve a vector represented as a directed line segment into its vertical and horizontal components

Sample problem: A cable exerts a force of 558 N at an angle of 37.2° with the horizontal. Resolve this force into its vertical and horizontal components.



1.4 represent a vector as a directed line segment, given its vertical and horizontal components (e.g., the displacement of a ship that travels 3 km east and 4 km north can be represented by the vector with a magnitude of 5 km and a direction of N 36.9° E)

1.5 determine, through investigation using a variety of tools (e.g., graph paper, technology) and strategies (i.e., head-to-tail method; parallelogram method; resolving vectors into their vertical and horizontal components), the sum (i.e., resultant) or difference of two vectors

1.6 solve problems involving the addition and subtraction of vectors, including problems arising from real-world applications (e.g., surveying, statics, orienteering)

Sample problem: Two people pull on ropes to haul a truck out of some mud. The first person pulls directly forward with a force of 400 N, while the other person pulls with a force of 600 N at a 50° angle to the first person along the horizontal plane. What is the resultant force used on the truck?

2. Solving Problems Involving Geometry

By the end of this course, students will:

2.1 gather and interpret information about real-world applications of geometric shapes and figures in a variety of contexts in technology-related fields (e.g., product design, architecture), and explain these applications (e.g., one reason that sewer covers are round is to prevent them from falling into the sewer during removal and replacement)

Sample problem: Explain why rectangular prisms are often used for packaging.

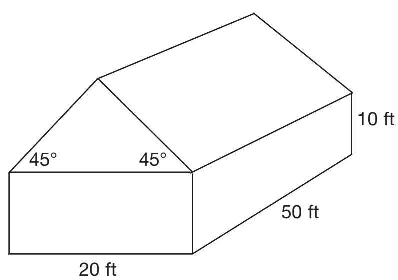
2.2 perform required conversions between the imperial system and the metric system using a variety of tools (e.g., tables, calculators, online conversion tools), as necessary within applications

2.3 solve problems involving the areas of rectangles, parallelograms, trapezoids, triangles, and circles, and of related composite shapes, in situations arising from real-world applications

Sample problem: Your company supplies circular cover plates for pipes. How many plates with a 1-ft radius can be made from a 4-ft by 8-ft sheet of stainless steel? What percentage of the steel will be available for recycling?

2.4 solve problems involving the volumes and surface areas of spheres, right prisms, and cylinders, and of related composite figures, in situations arising from real-world applications

Sample problem: For the small factory shown in the following diagram, design specifications require that the air be exchanged every 30 min. Would a ventilation system that exchanges air at a rate of 400 ft^3/min satisfy the specifications? Explain.



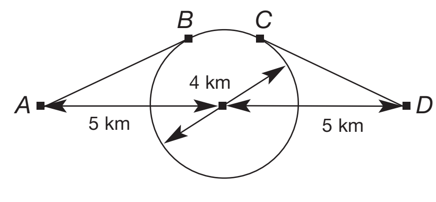
3. Solving Problems Involving Circle Properties

By the end of this course, students will:

3.1 recognize and describe (i.e., using diagrams and words) arcs, tangents, secants, chords, segments, sectors, central angles, and inscribed angles of circles, and some of their real-world applications (e.g., construction of a medicine wheel)

3.2 determine the length of an arc and the area of a sector or segment of a circle, and solve related problems

Sample problem: A circular lake has a diameter of 4 km. Points *A* and *D* are on opposite sides of the lake and lie on a straight line through the centre of the lake, with each point 5 km from the centre. In the route *ABCD*, *AB* and *CD* are tangents to the lake and *BC* is an arc along the shore of the lake. How long is this route?



3.3 determine, through investigation using a variety of tools (e.g., dynamic geometry software), properties of the circle associated with chords, central angles, inscribed angles, and tangents (e.g., equal chords or equal arcs subtend equal central angles and equal inscribed angles; a radius is perpendicular to a tangent at the point of tangency defined by the radius, and to a chord that the radius bisects)

Sample problem: Investigate, using dynamic geometry software, the relationship between the lengths of two tangents drawn to a circle from a point outside the circle.

3.4 solve problems involving properties of circles, including problems arising from real-world applications

Sample problem: A cylindrical metal rod with a diameter of 1.2 cm is supported by a wooden block, as shown in the following diagram. Determine the distance from the top of the block to the top of the rod.